

Independence of Parallel Postulate

May 14, 2013

1 Consistency of hyperbolic geometry

For thousands of years people believe and continue to believe that there is no contradiction in Euclidean geometry, that is, Euclidean geometry is consistent. How about hyperbolic geometry? Is there any contradiction in hyperbolic geometry?

Applied geometry (engineering) is about lines that we can draw. Pure geometry is about ideal lines which are concepts, not physical objects. Physical experiment can test the validity of physical systems, but cannot verify pure geometry. The only experiment that can perform on ideal lines are thought-experiments.

Can we conceive of a non-Euclidean geometry? Kant said no at his time (any geometry other than Euclidean geometry is inconceivable). Is hyperbolic geometry consistent? Nowadays this is referred to as a question in metamathematics, that is, a question outside of a mathematical system about the system itself. The question is not about lines or points or other geometric entities; it is about the *whole system* of hyperbolic geometry.

Metamathematical Theorem. If Euclidean geometry is consistent, then so is hyperbolic geometry.

To prove Metamathematical Theorem, we have to ask ourselves, What is a “line” in hyperbolic geometry – in fact, what is the hyperbolic plane? The honest answer is that we don’t know; it is just a formal system about undefined terms such as “points, lines” satisfying certain relations such as “betweenness, congruence, continuity” and “hyperbolic parallelism.” Then how shall we visualize hyperbolic geometry? In mathematics, as in any field of research, posing the right question is just as important as finding answers.

We have seen that the Euclidean parallel postulate is independent of the incidence axioms by exhibiting three-point and five-point models of incidence geometry that are not Euclidean. We want to know whether the parallel postulate is independent of a much larger system of axioms, namely, the neutral geometry. Here we can show that it is, by the same method – by exhibiting models for hyperbolic geometry.

2 Beltrami-Klein model (Klein model)

We fix once and all a circle γ in the Euclidean plane with center O and radius OR . The interior of γ is the set of points X such that $OX < OR$. A **chord** of γ is a segment AB in the Euclidean plane joining two endpoints A, B on γ . The open segment of chord AB , denoted (AB) , is called an **open chord** of γ . In Klein model points and lines are the interior points and open chords of γ respectively, and the incidence relation between points and lines are belongness. Poin P lies in line (AB) means that $P \in \overline{AB}$ in the Euclidean plane and $A * P * B$. See Figure 1

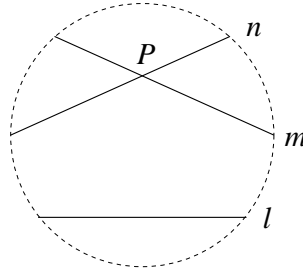


Figure 1: Klein model

3 Poincaré model

A disk model due to Henri Poincaré (1854-1912) also represents a hyperbolic plane whose points are **interior points** of a Euclidean circle γ , whose lines are those **open circular arcs orthogonal to γ** , and the incidence is the belongness of points and open circular arcs.

The interpretation of **congruence for segments** in the Poincaré model is complicated, being based on a way of measuring length that is different from the usual Euclidean way. **Congruence of angles** has the usual Euclidean meaning and it is the main advantage of the Poincaré model over the Klein model. Specifically, if two directed circular arcs intersect at a point A , the number of degrees in the angle they make is by definition the number of degrees in the angle between their tangent rays at A ; if one directed circular arc intersects an ordinary ray at A , the number of degrees in the angle they make is by definition the number of degrees in the angle between the tangent ray and the ordinary ray at A .

Two Poincaré lines are **parallel** if they have no point in common. In Poincaré model all axioms of hyperbolic geometry are translated into statements in Euclidean geometry. Hence the Poincaré model furnishes a proof that if Euclidean geometry is consistent, so is hyperbolic geometry.

The interior upper half-plane also serves as a model for hyperbolic plane, where points are ordinary points in the open upper half-plane and lines are those rays perpendicular to the x -axis and semicircle orthogonal to the x -axis.

4 Model of hyperbolic plane from physics

Consider the hyperboloid Σ defined by

$$x^2 + y^2 - t^2 = -1,$$

which can be thought as “sphere” centered at the origin $O = (0, 0, 0)$ with radius $\iota = \sqrt{-1}$. Let Π denote the plane $t = 0$ and Δ the disk of Π with center O and unit radius. The component

$$\Sigma := \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 - t^2 = -1, t > 0\}$$

can serve as a model for hyperbolic plane, where points are ordinary points of Σ and lines are nonempty intersections of Σ and Euclidean planes through O .

5 Inversion in circles

Proposition 5.1. *Given a circle γ of radius r with center O in the Euclidean plane, and a chord T_1T_2 not to be a diameter of γ . Let P be the intersection of the tangent lines of γ at*

T_1, T_2 , called the **pole** of T_1T_2 with respect to γ . Let P' be the middle point of T_1T_2 , and M the middle point of segment OP . Then

- (a) $O * P' * P$ and $|OP| \cdot |OP'| = r^2$.
- (b) $|PT_1| = |PT_2|$, $|OM| = |MP| = |MT_1| = |MT_2|$.

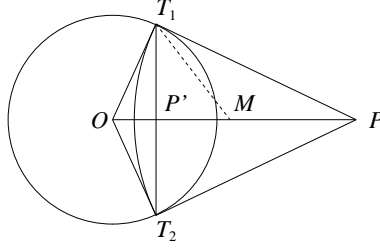


Figure 2: Inversion about circle

Proof. (a) It is clear that P' is between O and P , that is, $O * P' * P$. Note that $\angle OP'T_1$ and $\angle T_1P'P$ are right angles, and the right triangles $\triangle OP'T_1, \triangle OT_1P$ have the common angle $\angle P'OT_1 = \angle T_1OP$. We have $\triangle OP'T_1 \sim \triangle OT_1P$. Then the corresponding sides of $\triangle OP'T_1, \triangle OT_1P$ are proportional. Hence $|OP'| : |OT_1| = |OT_1| : |OP|$, where $|OT_1| = r$, that is, $|OP| \cdot |OP'| = r^2$.

(b) Clearly, $|PT_1| = |PT_2|$. Since $\triangle OT_1P$ and $\triangle OT_2P$ are right triangles, the points O, T_1, P, T_2 lie on a common circle of diameter OP . So $OM \cong MP \cong MT_1 \cong MT_2$. \square

Definition 1. Let γ be a circle of radius r with center O in the Euclidean plane. For each point $P \neq O$, its **inverse** with respect to γ is the unique point P' on the ray $r(O, P)$ such that

$$|OP| \cdot |OP'| = r^2.$$

Let \mathbf{E} denote the Euclidean plane and $\tilde{\mathbf{E}} = \mathbf{E} \cup \{\infty\}$. The map from $\tilde{\mathbf{E}}$ to itself by $P \mapsto P', O \mapsto \infty, \infty \mapsto O$ is called the **inversion** in γ .

Proposition 5.2. Let P, P' be inverses each other about the inversion in circle γ with center O . Then

- (a) $P' = P \Leftrightarrow P \in \gamma$.
- (b) P is inside $\gamma \Leftrightarrow P'$ is outside γ .

Proof. Trivial by definition. \square

Lemma 2 (Circle-Cut Product Property). Let P be a point not on a circle γ . Given three lines passing through P ; the first one intersects γ in a pair of points A_1, A_2 , the second intersects γ in a another pair of points B_1, B_2 , and the third is tangent to γ at a point D ; see Figure 3. Then

$$|PA_1| \cdot |PA_2| = |PB_1| \cdot |PB_2| = |PD|^2.$$

The constant is called the **power** of P with respect to γ .

Proof. Since angles that are inscribed in a circle and subtended the same arc are congruent, then $\angle A_1A_2B_1 \cong \angle A_1B_2B_1$ and $\angle A_2A_1B_2 \cong \angle A_2B_1B_2$. Thus $\triangle PA_2B_1 \sim \triangle PB_2A_1$. See Figure 3. So the corresponding sides are proportional, that is, $|PA_1| : |PB_1| = |PB_2| : |PA_2|$.

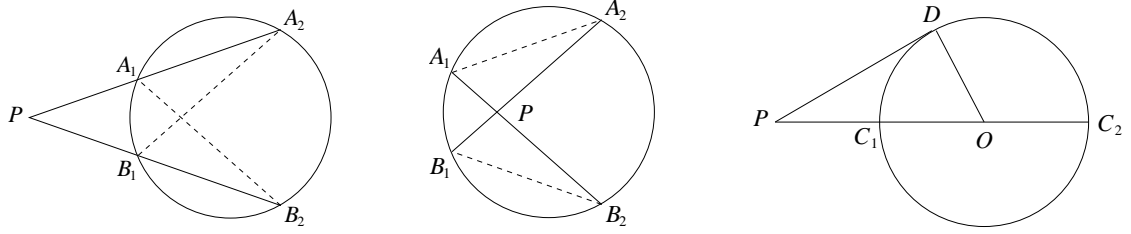


Figure 3: Circle cut

Let line \overline{PO} intersect γ at in a pair of points C_1, C_2 with $P * C_1 * O * C_2$. Then $\triangle PDO$ is a right triangle. By the Pythagorean theorem we have

$$\begin{aligned}
 |PD|^2 &= |OP|^2 - |OD|^2 \\
 &= (|OP| - |OD|)(|OP| + |OD|) \\
 &= (|OP| - |OC_1|)(|OP| + |OC_2|) \\
 &= |PC_1| \cdot |PC_2|.
 \end{aligned}$$

□

Proposition 5.3 (Orthogonality and Cocyclicity of Inverse Points). *Given a circle γ of radius r with center O , and two points P, P' inverse each other with respect to γ . Let δ be a circle through P , intersecting γ at T_1, T_2 . Then γ, δ meet orthogonally if and only if δ passes through P' .*

Proof. “ \Rightarrow ”: Let δ intersect γ orthogonally at T_1, T_2 . Then the tangents to δ at T_1, T_2 pass through O . So O lies outside δ , and ray $r(O, P)$ cuts δ at a point Q . The Circle-Cut Product Property implies $|OP| \cdot |OQ| = |OS_1|^2 = r^2$. Hence Q is the inverse of P with respect to γ , that is, $P' = Q \in \delta$. See Figure 4.

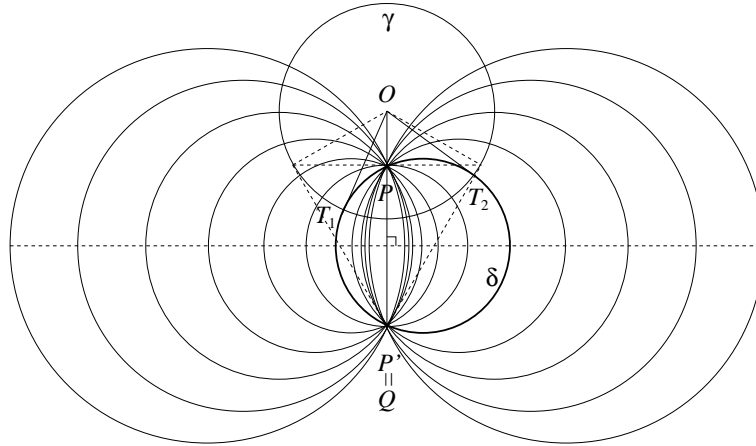


Figure 4: Poincaré lines through a point P .

“ \Leftarrow ”: Let δ pass through P' . Then the center C of δ lies on the perpendicular bisector of PP' . Note that either $O * P * P'$ or $O * P' * P$. Clearly, point O is outside δ . Then through O there exist two tangents to δ at T_1^*, T_2^* . Note that T_1^*, T_2^* are not assumed on γ . On the one hand, $|OP| \cdot |OP'| = |OT_1^*|^2 = |OT_2^*|^2$ by the Cut-Circle Product Property. On the other hand, $|OP| \cdot |OP'| = r^2$ since P, P' are inverses each other with respect to γ . Then $|OT_1^*| = |OT_2^*| = r$. So $T_1^* = T_1$ and $T_2^* = T_2$. Of course, the tangent ray $r(T_1, O)$ of δ at T_1 is orthogonal to γ at T_1 . This means that δ is orthogonal to γ . See Figure 4. □

Corollary 5.4 (Inversions in two orthogonal circles map each of them onto itself). *A circle δ is orthogonal to a circle $\gamma \Leftrightarrow \delta$ is mapped onto itself by inversion in γ .*

Proof. Let P be a point on δ and P' its inverse in γ . Then O, P, P' are collinear. If δ is orthogonal to γ , then P' lies on δ by Proposition 5.3.

Conversely, if the inversion in γ maps δ onto itself, of course δ passes through P, P' , then δ is orthogonal to γ by Proposition 5.3. \square

Given a circle γ ; the open disk bounded by γ is called a **Poincaré disk**. Given two points A, B inside γ , there exists a unique circle σ through A, B and orthogonal to γ . The circular arc of σ inside γ is called a **Poincaré line** or **P -line** (= line in hyperbolic geometry) through points A, B in the Poincaré disk bounded by γ . (**Axiom 1-3** of incidence geometry are satisfied.) The circular arc between A and B on the P -line through A, B is called a **Poincaré segment** or **P -segment**.

Definition 3. Let A, B be two points inside a circle γ , and P, Q the endpoints of the hyperbolic line through A, B . The **cross-ratio** of A, B is defined as

$$(AB, PQ) := \frac{|AP| \cdot |BQ|}{|AQ| \cdot |BP|}.$$

The **hyperbolic distance** between two points A, B is defined as

$$d(A, B) := |\log(AB, PQ)|.$$

Notice that distance does not depend on the the order of A, B . In fact, if $(AB, PQ) = x$ then $(BA, PQ) = 1/x$, so

$$d(B, A) = |\log(1/x)| = |-\log x| = |\log x| = d(A, B).$$

Two P -segments AB, CD are said to be **Poincaré-congruent** if $d(A, B) = d(C, D)$. With this interpretation, **Congruence axiom 2** is immediately verified, for the relation of equal hyperbolic distance is an equivalence relation.

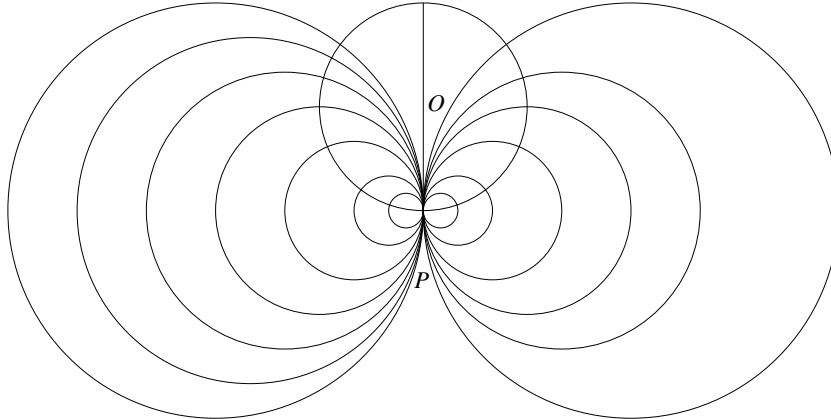


Figure 5: Limiting parallel Poincaré lines.

Let A, B, C be points on a P -line inside the Poincaré disk bounded by γ , having end points P, Q on γ . We may assume the order $P * A * B * C * Q$ on the P -line. See Figure 6. We have cross-ratio $(AB, PQ) = (|AP| \cdot |BQ|) / (|AQ| \cdot |BP|) < 1$. Likewise, $(AC, PQ) < 1$

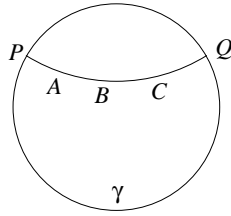


Figure 6: Poincaré line inside a disk.

and $(BC, PQ) < 1$. Their logs are negative and

$$\begin{aligned}
 d(A, B) + d(B, C) &= -\log(AB, PQ) - \log(BC, PQ) \\
 &= -\log[(AB, PQ) \cdot (BC, PQ)] \\
 &= -\log\left(\frac{|AP|}{|AQ|} \cdot \frac{|BQ|}{|BP|} \cdot \frac{|BP|}{|BQ|} \cdot \frac{|CQ|}{|CP|}\right) \\
 &= -\log\left(\frac{|AP|}{|AQ|} \cdot \frac{|CQ|}{|CP|}\right) \\
 &= d(A, C).
 \end{aligned}$$

Congruence axiom 3 (addition rule) is satisfied.

Definition 4. Let O be a point and k a positive number. The **dilation with center O and ratio k** is the transformation of the Euclidean plane that fixes O and maps each point $P \neq O$ to a unique point P^* on ray $r(O, P)$ such that $OP^* \cong k \cdot OP$.

Dilation with center O and ratio k maps lines to lines, circles to circles, and preserves angles. In fact, choose rectangular coordinates so that O is the origin. The dilation is the map $(x, y) \mapsto (x', y') = k(x, y)$. Then $x = x'/k, y = y'/k$. For line $l : ax + by = c$, its image under the dilation is $ax'/k + by'/k = c$, which is the line $ax' + by' = kc$. For a circle $\gamma : (x - a)^2 + (y - b)^2 = r^2$, its image is $(x'/k - a)^2 + (y'/k - b)^2 = r^2$, which is the circle $(x' - ka)^2 + (y' - kb)^2 = (kr)^2$.

Proposition 5.5. Let γ be a circle of radius r and center O . Let δ be a circle with center C such that O is outside δ . Set $k = r^2/p$, where p is the power of O with respect to δ .

(a) Then the image δ' of δ under inversion in γ is a circle δ^* , obtained from δ by the dilation with center O and ratio k .

(b) If $P \in \delta$ and P' its inverse in γ , then the tangent t to δ at P and the tangent t' to δ' at P' are symmetric about the perpendicular bisector of segment PP' .

Proof. (a) Let P be a point on δ and P' its inverse with respect to γ . Let ray $r(O, P)$ meet circle δ at another point Q (with $Q = P$ when $r(O, P)$ is tangent to δ). Then

$$\frac{|OP'|}{|OQ|} = \frac{|OP'|}{|OQ|} \cdot \frac{|OP|}{|OP|} = \frac{r^2}{p} = k.$$

Since $P', Q \in r(O, P)$, we have $OP' = k \cdot OQ$, that is, the vector $\overrightarrow{OP'}$ is the dilation of the vector \overrightarrow{OQ} with ratio k . This means that P' is on the circle δ^* obtained from δ by the dilation with center O and ratio k . So $\delta' = \delta^*$. The center C^* of δ^* is the same dilation of the center C of δ , that is, $C^* \in r(O, C)$ and $OC^* = k \cdot OC$.

(b) Since OP' is the dilation of OQ , the tangent t' of δ' at P' is parallel to the tangent t'' of δ at Q . Let t meet t'' at R and meet t' at S . Then $\angle PP'S \cong \angle PQR \cong \angle P'PS$. So $\triangle PP'S$ is an isosceles triangle. Hence t, t' are symmetric about the perpendicular bisector of PP' . See Figure 7. \square

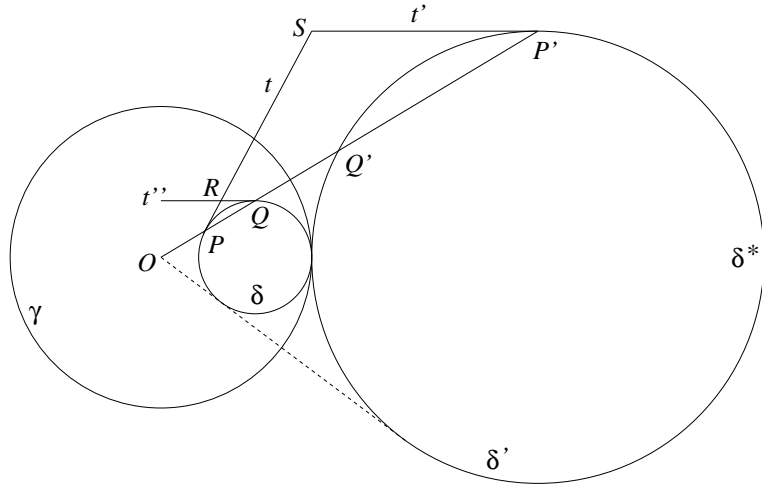


Figure 7: Image of a circle by inversion in γ is a circle.

Lemma 5. *Given a circle γ with center O , and two points P, Q such that O, P, Q are not collinear. Let P', Q' be inverses of P, Q in γ respectively. Then $\triangle OPQ \sim \triangle OQ'P'$.*

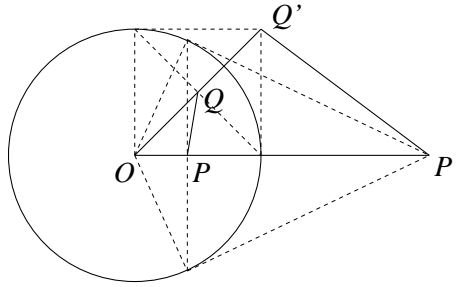


Figure 8: Image of triangle under inversion in circle is similar.

Proof. Since $P' \in r(O, P)$ and $Q' \in r(O, Q)$, the triangles $\triangle OPQ, \triangle OQ'P'$ have the common angle $\angle POQ = \angle P'OQ'$. Note that $|OP| \cdot |OP'| = r^2 = |OQ| \cdot |OQ'|$, that is,

$$|OP| : |OQ'| = |OQ| : |OP'|.$$

This means that $\triangle OPQ \sim \triangle OQ'P'$. □

Proposition 5.6. *Given a circle γ with center O . Let l be a line not passing through O . Then the image of l under inversion in γ is a circle δ through O , but having O removed, and its diameter through O is perpendicular to l .*

Proof. Let OP be the segment perpendicular to l with foot P on l . Let X be an arbitrary point on l . Let P', X' be the inverse points of P, X in γ respectively. Note that $P' \in r(O, P)$ and $X' \in r(O, X)$. Then $\triangle OPX \sim \triangle OX'P'$. So $\angle OX'P' \cong \angle OPX$ is a right angle. Thus X' is on the circle δ with diameter OP' . □

Corollary 5.7. *Given a circle γ with center O . Let δ be a circle passing through O . Then the image of $\delta - \{O\}$ under inversion in γ is a line l not through O , and l is parallel to the tangent of δ at O .*

Proof. Trivial. □

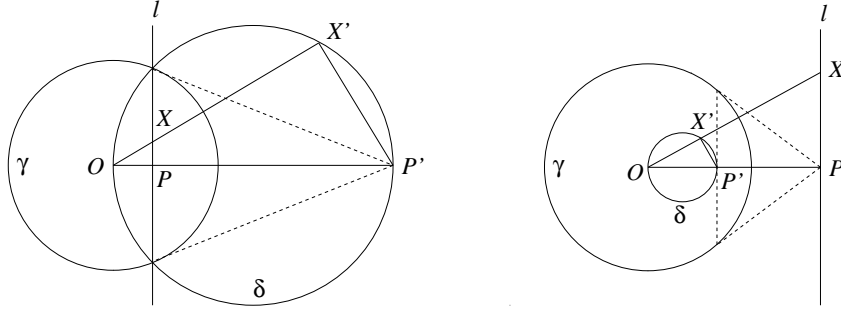


Figure 9: Image of a line under inversion in γ is a circle through center of γ .

Proposition 5.8. *The directed angle of two circles at their intersection point are preserved by inversion in circle.*

Proof. Let γ be a circle with center O . Let σ, δ be two circles intersecting at a point $P \neq O$. Let σ', δ', P' be the images of σ, δ, P under inversion in γ respectively. Given tangent rays s of σ and t of δ at P , and tangent rays s' of σ' and t' of δ' at P' . Then rays s, s' are symmetric about the perpendicular bisector of PP' ; so are the rays t, t' . Hence the angle between s' and t' are congruent to the angle between s and t . \square

Proposition 5.9. *Given a circle γ with center O . Let A, B, C, D be four points distinct from O , and A', B', C', D' their inverses in γ respectively. Then the cross-ratio is preserved by inversion in γ , that is,*

$$(AB, CD) = (A'B', C'D').$$

Proof. Recall $(AB, CD) := (|AC| \cdot |BD|) / (|AD| \cdot |BC|)$. Consider similar triangles $\triangle OAC \sim \triangle OC'A'$ and $\triangle OAD \sim \triangle OD'A'$. We have

$$|AC| : |OA| = |A'C'| : |OC'|, \quad |AD| : |OA| = |A'D'| : |OD'|.$$

Then

$$\frac{|AC|}{|AD|} = \frac{|A'C'|}{|A'D'|} \cdot \frac{|OD'|}{|OC'|}. \quad (1)$$

Likewise

$$\frac{|BD|}{|BC|} = \frac{|B'D'|}{|B'C'|} \cdot \frac{|OC'|}{|OD'|}. \quad (2)$$

Multiplying (1) and (2), we obtain

$$(AB, CD) = \frac{|AC|}{|AD|} \cdot \frac{|BD|}{|BC|} = \frac{|A'C'|}{|A'D'|} \cdot \frac{|B'D'|}{|B'C'|} = (A'B', C'D').$$

\square

Proposition 5.10. *Given two orthogonal circles γ, δ with centers O, C and radii r, d respectively. Let σ be a circle and l a line, both are orthogonal to γ . Let σ', l' be respectively the images of σ, l under inversion in δ .*

- (a) *Then the inversion in δ maps γ onto γ , and maps the interior of γ onto itself.*
- (b) *If σ does not pass through C , then σ' is a circle orthogonal to γ . If σ passes through C , then σ' is a line passing through the center O of γ .*
- (c) *If l does not pass through C , then l' is a circle orthogonal to γ . If l passes through C , then $l = \overline{OC}$ and $l' = l$.*
- (d) *Inversion in δ preserves incidence, betweenness, and congruence in the sense of the Poincaré disk model inside γ .*

Proof. (a) The former part is implied by the fact that circle orthogonal to an inversion circle is fixed by the inversion, namely, Corollary 5.4. For the latter part, let P be a point inside γ and P' its inverse in δ . Let $r(C, P)$ intersect γ at points Q, Q' . See Figure 10. Since γ is

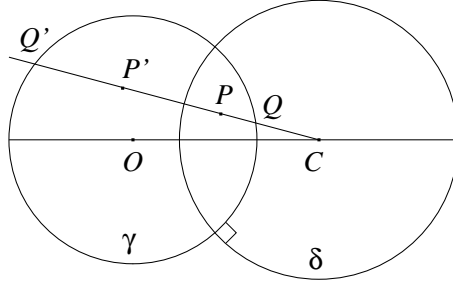


Figure 10: Inversion in δ maps interior of γ to itself.

orthogonal to δ , the point Q' must be the inverse of Q in δ by Proposition 5.3. Then

$$|CP| \cdot |CP'| = |CQ| \cdot |CQ'| = d^2.$$

Note that C, P, P', Q, Q' are collinear and $Q * P * Q'$. We may assume $C * Q * P * Q'$ without loss of generality. Then $|CQ| < |CP| < |CQ'|$. Thus $d^2/|CQ| > d^2/|CP| > d^2/|CQ'|$, which is the same as $|CQ'| > |CP'| > |CQ|$. This means that $C * Q * P * Q'$. So P' is inside γ .

(b),(c) Trivial.

(d) The circular arc of σ inside γ is a P -line, and is mapped to the circular arc of σ' inside γ , which is another P -line. Given a P -line through A, B inside γ with end points P, Q on γ . The P -line through the images A', B' of A, B by inversion in δ have end points P', Q' on γ . Since $(AB, PQ) = (A'B', P'Q')$, we see that

$$d(A, B) = |\log(AB, PQ)| = |\log(A'B', P'Q')| = d(A', B').$$

So P -segment AB is congruent to P -segment $A'B'$ hyperbolically. Let $A * B * D$. Then

$$d(A', D') = d(A, D) = d(A, B) + d(A', B') = d(A', B') + d(B', D')$$

So B' is between A' and D' . □

Verification of Congruence Axiom 6 (SAS). Given P -triangles $\triangle ABC$ and $\triangle XYZ$ inside a circle γ of radius r with center O , such that $d(A, B) = d(X, Y)$, $\angle A \cong \angle X$, and $d(A, C) = d(X, Z)$. We need to show that $\triangle ABC$ and $\triangle XYZ$ are P -congruent, that is, the corresponding sides have the same hyperbolic lengths and the corresponding angles have the same measures.

Case 1. $A = X = O$ (the center of γ).

Then $d(O, B) = d(O, Y)$, $d(O, C) = d(O, Z)$, and $\angle BOC \cong \angle YOZ$. Let P, Q be end points of P -line through points O, B and $P * B * O * Q$. Then

$$d(O, B) = \log(|OP| \cdot |BQ| / |OQ| \cdot |BP|) = \log[(r + |OB|) / (r - |OB|)].$$

It follows that $|OB| = r(e^{d(O, B)} - 1) / (e^{d(O, B)} + 1)$. Hence $|OB| = |OY|$, $|OC| = |OZ|$, and $\angle BOC \cong \angle YOZ$. There exists a unique rigid motion (rotation about O , combined with a reflection about a diameter of γ if necessary) such that the images of B, C under the rigid motion coincide with Y, Z respectively. So P -triangles $\triangle OBC$ and $\triangle OYZ$ are P -congruent, since rotation about O and reflection about diameter of γ preserve hyperbolic length and angle.

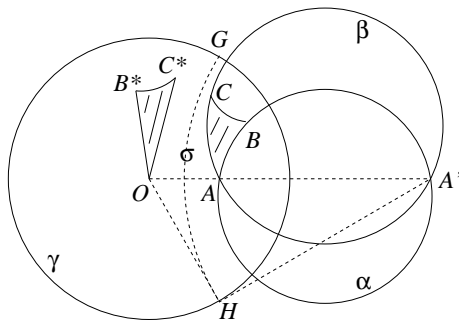


Figure 11: Image of triangle under inversion in circle is similar.

Case 2. General case is reduced to *Case 1*.

Let α, β be the circle orthogonal to γ through A, B , and β the circle orthogonal to γ through A, C . Then α, β meet at A' outside γ , which is the inverse of A in γ by Proposition 5.3. Then O, A, A' are collinear and $O * A * A'$. Let σ be the circle of radius s with center A' , where $s^2 = |A'A| \cdot |A'O|$. Then A is inside σ , O is outside σ , and O, A are inverses each other in σ by definition of inversion in circle. So σ, γ intersect at two points G, H . Since $|AA'| = |OA'| - |OA|$, we have

$$s^2 = (|OA'| - |OA|) \cdot |OA'| = |OA'|^2 - |OA| \cdot |OA'| = |OA'|^2 - r^2.$$

Thus $|OA'|^2 = r^2 + s^2 = |OH|^2 + |A'H|^2$. The Pythagorean theorem implies that $\triangle OA'H$ is a Euclidean right triangle with right angle at H . So σ is orthogonal to γ . See Figure 11.

Now the inversion in σ maps inside of γ onto itself, and in particular maps A to O . So the inversion in σ maps the P -triangle $\triangle ABC$ onto a P -triangle $\triangle OB^*C^*$. Clearly, P -triangles $\triangle ABC$ and $\triangle OB^*C^*$ are P -congruent. Likewise, P -triangle $\triangle XYZ$ can be mapped onto a P -triangle $\triangle OY^*Z^*$ by an inversion in a circle orthogonal to γ , and $\triangle OY^*Z^*$ is P -congruent to $\triangle XYZ$. Hence $\triangle OB^*C^*$ and $\triangle OY^*Z^*$ are P -triangles, satisfying $d(O, B^*) = d(O, Y^*)$, $\angle B^*OC^* \cong \angle Y^*OZ^*$, and $d(O, C^*) = d(O, Z^*)$. So $\triangle OB^*C^*$ and $\triangle OY^*Z^*$ are P -congruent by Case 1. Therefore $\triangle OBC$ and $\triangle OYZ$ are P -congruent.

We then have the following theorem.

Theorem 5.11. *Two P -triangles in the Poincaré disk bounded by a circle γ are P -congruent if and only if they can be mapped onto each other by a rotation about the center O of γ , and a succession of inversions in circles orthogonal to γ and in lines through the center of γ .*

Theorem 5.12. *Let d be Poincaré distance of point P to P -line l not through P inside Poincaré disk bounded by circle γ of radius r . Let $\Pi(d)$ denote the number of radians in the angle of parallelism of limiting parallel rays through P . Then*

$$e^{-d} = \tan[\Pi(d)/2].$$

Proof. Without loss of generality we may assume that l is the diameter of γ and PO is the segment perpendicular to l with foot O on l . Let α be the angle between Euclidean ray $r(P, Q)$ and the limiting P -line δ parallel to l . Let t be the tangent to δ at P . Then t meets l at R . Let S be the common end point of l and δ . Note that both l, δ are tangent to δ . So $\angle RSP \cong \angle RPS$. Set $\alpha := \angle QPR = \Pi(d)$ and $\beta := \angle RSP$. Then $\alpha + 2\beta = \pi/2$, that is, $\beta = \pi/4 - \alpha/2$. Note that $|OP| = r \tan \beta$. Then

$$d = d(O, P) = \left| \log \frac{r(r - |OP|)}{r(r + |OP|)} \right| = \log \frac{r + |OP|}{r - |OP|} = \log \frac{1 + \tan \beta}{1 - \tan \beta},$$

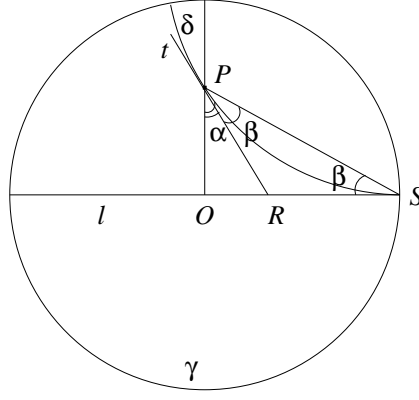


Figure 12: Angle of limiting parallel rays.

that is, $e^d = (1 + \tan \beta)/(1 - \tan \beta)$. Since $e^{i\theta} = \cos \theta + i \sin \theta$, then

$$e^{i(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2),$$

$$e^{i(\theta_1 + \theta_2)} = e^{i\theta_1} e^{i\theta_2} = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2).$$

Thus

$$\tan(\theta_1 + \theta_2) = \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} = \frac{\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2}{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2} = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}.$$

So $\tan \beta = \tan(\pi/4 - \alpha/2) = (1 - \tan(\alpha/2))/(1 + \tan(\alpha/2))$. Hence $e^d = \frac{2}{2 \tan(\alpha/2)}$, that is, $e^{-d} = \tan(\Pi(d)/2)$. \square

6 Isomorphism between Poincaré model and Klein model

Let Σ be a sphere in Cartesian 3-dimensional Euclidean space \mathbb{R}^3 , given by the equation

$$x_1^2 + x_2^2 + x_3^2 = r^2.$$

Let Δ be the open disk on the plane $x_3 = 0$, that is,

$$\Delta = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 < r^2\},$$

whose boundary is the circle

$$\gamma = \{(x_1, x_2, 0) : x_1^2 + x_2^2 = r^2\}.$$

We consider the map $F : \Delta \rightarrow \Delta$, induced by the stereographic projection through the lower hemisphere

$$\Sigma^- = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = r^2, x_3 < 0\}.$$

Simplicity let us use complex variable $z = x_1 + ix_2$ to denote point (x_1, x_2) in Δ . Then F is determined by

$$\frac{|F(z)| - |z|}{|z|} = \frac{\sqrt{r^2 - |F(z)|^2}}{r}.$$

Squaring both sides, we have $|F(z)|/|z| - 1)^2 = 1 - |F(z)|^2/r^2$, that is,

$$|F(z)| \cdot (|F(z)|(1/r^2 + 1/|z|^2) - 2/|z|) = 0.$$

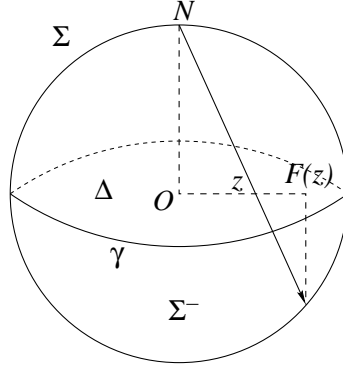


Figure 13: Stereographic projection

Since $F(z) \neq 0$ when $z \neq 0$, we further have $|F(z)| = 2r^2|z|/(r^2 + |z|^2)$, that is,

$$F(z) = 2r^2z/(r^2 + |z|^2).$$

Writing in rectangular coordinates, we have

$$F(x_1, x_2) = \frac{2r^2}{r^2 + x_1^2 + x_2^2}(x_1, x_2).$$

Theorem 6.1. *The stereographic projection is an isomorphism from the Poincaré model to the Klein model. More specifically, given a circle δ orthogonal to γ , intersecting γ at P, Q . If A is a point on the arc of δ from P to Q inside γ , then the ray $r(O, A)$ meets the chord PQ of γ at $F(A)$.*

Proof. Let C be the center of δ with coordinates (c_1, c_2) . Since γ, δ meet orthogonally at P, Q , the points O, P, Q, C lie on a circle σ with diameter OC , having center $(\frac{c_1}{2}, \frac{c_2}{2})$ and radius $\frac{1}{2}\sqrt{c_1^2 + c_2^2}$. The equation of σ is

$$\left(x_1 - \frac{c_1}{2}\right)^2 + \left(x_2 - \frac{c_2}{2}\right)^2 = \frac{c_1^2 + c_2^2}{4}, \quad \text{i.e.,} \quad x_1^2 + x_2^2 = c_1x_1 + c_2x_2.$$

Then P, Q are the solutions of the system

$$\begin{cases} x_1^2 + x_2^2 = c_1x_1 + c_2x_2 \\ x_1^2 + x_2^2 = r^2 \end{cases}, \quad \text{i.e.,} \quad \begin{cases} c_1x_1 + c_2x_2 = r^2 \\ x_1^2 + x_2^2 = r^2 \end{cases}.$$

Clearly, line $c_1x_1 + c_2x_2 = r^2$ passes through P, Q ; it must be the equation of line \overline{PQ} . Note

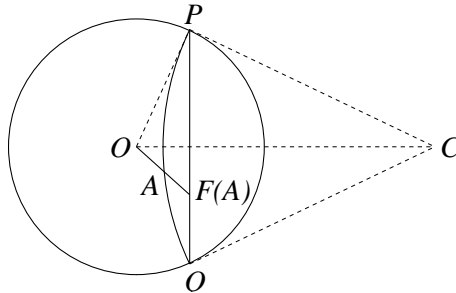


Figure 14: The map between Poincaré model and Klein model.

that $\angle OPC$ is a right angle; the Pythagorean theorem implies

$$|CP|^2 = |OC|^2 - |OP|^2 = c_1^2 + c_2^2 - r^2.$$

The equation $(x_1 - c_1)^2 + (x_2 - c_2)^2 = |CP|^2$ of δ becomes

$$x_1^2 + x_2^2 = 2c_1x_1 + 2c_2x_2 - r^2.$$

Now if $A = (a_1, a_2)$ lies on δ and $F(A) = (b_1, b_2)$ is the image under F , then

$$a_1^2 + a_2^2 + r^2 = 2c_1a_1 + 2c_2a_2 > 0,$$

and subsequently,

$$b_j = \frac{2r^2a_j}{r^2 + a_1^2 + a_2^2} = \frac{r^2a_j}{c_1a_1 + c_2a_2}, \quad j = 1, 2.$$

Hence

$$c_1b_1 + c_2b_2 = r^2,$$

which means that (b_1, b_2) lies on the line \overline{PQ} . □

Proposition 6.2. *Given a point P inside a circle γ with center O and radius r , and its inverse P' in γ . Let AB be a chord of γ with midpoint P . Then the circle δ with center P' and radius $P'A$ is orthogonal to γ . Moreover, the inverse of P in δ is O .*

Proof. Note that $|OA| \cdot |OA'| = r^2$ and $r^2 - |OP|^2 = |AP|^2 = d^2 - |PP'|^2$. □

7 Philosophical Implications

7.1 What is the geometry of physical space

If Euclidean geometry is consistent, so is hyperbolic geometry. Then two geometries are equally consistent. Logically speaking, hyperbolic geometry deserves to be put on an equal footing with Euclidean geometry. Engineering and architecture are evidence that Euclidean geometry is extremely useful for ordinary measurement of distances that are not large. However, the representational accuracy of Euclidean geometry is less certain when deal with larger distances. Because of experimental error, a physical experiment can never prove conclusively that space is Euclidean – it can prove only that space is non-Euclidean.

According to Einstein, space and time are inseparable and the geometry of spacetime is affected by matter, so that light rays indeed curved by the gravitational attraction of masses. Space is no longer conceived of as an empty Newtonian box whose contours are unaffected by the rocks put into it. The problem is much more complicated than Euclidean or non-Euclidean – neither of the geometries is adequate for the present conception of space. This does not diminish the historical importance of Euclidean and non-Euclidean geometries. Einstein said, “To this interpretation of [hyperbolic] geometry I attached great importance, for should I not have acquainted with it, I never would have been able to develop the theory of relativity.”

Henri Poincaré said: “If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision The geometrical axioms are therefore neither synthetic a priori intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when experimental laws which have determined their adoption are only approximate. In other words, the axioms of geometry (I do not speak of those of arithmetic) are only definitions in disguise.”

We might think that Euclidean geometry is the most convenient – it is for ordinary engineering, but not for the theory of relativity. One school, which includes Newton, Helmholtz, Russel, and Whitehead, contends that space has an intrinsic metric or standard measurement. The other school, which includes Riemann, Poincaré, Clifford, and Einstein, contends that a metric stipulated by convention. The discussion can become very subtle.

7.2 What is mathematics about?

Geometry is not about light rays, but the path of a light ray is one possible physical interpretation of the undefined geometric term “line.” Bertrand Russell once said that “mathematics is the subject in which we do not know we are talking about nor what we say is true.” This is because certain primitive terms such as “point,” “line,” and “plane,” are undefined and could just as well be replaced with other terms without affecting the validity of results.

Gottlob Frege (1848-1925), who is considered the founder of modern mathematical logic, wrote to Hilbert: “I give the name of axioms to propositions which are true, but which are not demonstrated because their knowledge proceeds from a source which is not logical, which we may call space time. The truth of the axiom implies of course that they do not contradict each other. That is to me the criterion of truth and existence.”

Euclidean and hyperbolic geometries were equally consistent, so they “exist” and are both “true.” The discovery that Euclidean geometry was not “absolute truth” had a liberating effect on mathematicians, who now feel free to invent any set of axioms they wish and deduce conclusions from them. In fact, this freedom may account for the great increase in the scope and generality of modern mathematics. In a 1961 address, Jean Dieudonné remarked on Gauss’ discovery of non-Euclidean geometry: “[It] was a turning point of capital significance in the history of mathematics, marking the first step in a new conception of the relation between the real world and the mathematical notions supposed to account for it; with Gauss’ discovery, the rather naive point of view that mathematical objects were only “ideas” (in the Platonic sense) of sensory objects became untenable, and gradually gave way to a clearer comprehension of the much greater complexity of the question, wherein it seems to us today that mathematics and reality are almost completely independent, and their contacts more mysterious than ever.”

7.3 Controversy about foundation of mathematics

It would be misleading to say that mathematics is just a formal game played with symbols and having no broader significance. Mathematicians do not arbitrarily make up axioms and deduce their conclusions. Axioms must be lead to interesting and fruitful results. Of course, some axioms that appear uninteresting may turn out to have surprising consequences – this was the case with the hyperbolic axiom, which was virtually ignored during the lifetimes of Gauss, Bolyai, and Lobachevsky. If, however, axiom systems do not bear interesting results, they become neglected and eventually forgotten.

Arguing against the description of mathematics as a “formal game,” R. Courant and H. Robbins (in their fine book *What is Mathematics?*) insist that “a serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definition, rules and syllogisms, without motivation or goal.”

Hermann Weyl reamrked: “The construction of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and to

set up his axioms as he pleased. But the question is, will he get his fellow mathematicians interested in the constructs of his imagination?"

Leopold Kronecker said: "God created whole numbers – all else is manmade."

Axiom of Choice (AC): *Given sets S_i indexed by $i \in I$. There exist elements $a_i \in S_i$ for all $i \in I$.*

Continuum Hypothesis (CH): *No cardinal number between \aleph_0 (the cardinality of the set of positive integers) and c (the cardinal number of the set of real numbers).*

Kurt Gödel created a model of the other A-Z axioms in which both AC and CH were true; that demonstrated the impossibility of disproving them. In 1963, models were created in which either AC or CH or both were false. So AC and CH are independent of the other Z-F axioms and of each other. There exists an equally valid non-Cantorian set theory, just as there is an equally valid non-Euclidean geometry.

Gödel's Incompleteness Theorem: *There will always be valid statements that cannot be demonstrated from systems of axioms that are broad enough to include arithmetic. In other words, Gödel provided a formal demonstration of the inadequacy of formal demonstration.*